

MINIMUM DISTANCE AND THE MINIMUM WEIGHT CODEWORDS OF SCHUBERT CODES

SUDHIR R. GHORPADE AND PRASANT SINGH

ABSTRACT. We consider linear codes associated to Schubert varieties in Grassmannians. A formula for the minimum distance of these codes was conjectured in 2000 and after having been established in various special cases, it was proved in 2008 by Xiang. We give an alternative proof of this formula. Further, we propose a characterization of the minimum weight codewords of Schubert codes by introducing the notion of Schubert decomposable elements of certain exterior powers. It is shown that codewords corresponding to Schubert decomposable elements are of minimum weight and also that the converse is true in many cases. A lower bound, and in some cases, an exact formula, for the number of minimum weight codewords of Schubert codes is also given. From a geometric point of view, these results correspond to determining the maximum number of \mathbb{F}_q -rational points that can lie on a hyperplane section of a Schubert variety in a Grassmannian with its nondegenerate embedding in a projective subspace of the Plücker projective space, and also the number of hyperplanes for which the maximum is attained.

1. INTRODUCTION

Fix a prime power q and positive integers ℓ, m with $\ell \leq m$. Let \mathbb{F}_q denote the finite field with q elements and let V be a vector space over \mathbb{F}_q of dimension m . To the Grassmannian $G_{\ell, m}$ of all ℓ -dimensional linear subspaces planes of V , one can associate in a natural way a $[n, k]_q$ -code, i.e., a q -ary linear code of length n and dimension k , where

$$(1) \quad n = \begin{bmatrix} m \\ \ell \end{bmatrix}_q := \frac{(q^m - 1)(q^m - q) \cdots (q^m - q^{\ell-1})}{(q^\ell - 1)(q^\ell - q) \cdots (q^\ell - q^{\ell-1})} \quad \text{and} \quad k = \begin{pmatrix} m \\ \ell \end{pmatrix}.$$

This code is denoted by $C(\ell, m)$ and is called *Grassmann code*. The study of Grassmann codes goes back to the work of Ryan [16, 17, 19] in the late 1980's and was continued by Nogin [14] and several authors (see, e.g., [6, 7, 10, 5, 2] and the references therein). It is now known that Grassmann codes possess a number of

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interesting properties. For instance, their minimum weight is known and is given by the following beautiful formula of Nogin [14]:

$$(2) \quad d(C(\ell, m)) = q^\delta \quad \text{where} \quad \delta := \ell(m - \ell).$$

Furthermore, several of generalized Hamming weights are known, the automorphism group has been determined and is known to be fairly large, the duals of Grassmann codes have a very low minimum distance (namely, 3) and the minimum weight codewords of $C(\ell, m)^\perp$ generate $C(\ell, m)^\perp$. In fact, as the results of [2] show, Grassmann codes can be regarded as regular LDPC codes and are expected to be amenable to iterative decoding.

Schubert codes are a natural generalization of Grassmann codes and were introduced in [6] around the turn of the last century. These are linear codes $C_\alpha(\ell, m)$ associated to Schubert subvarieties $\Omega_\alpha(\ell, m)$ of the Grassmannian $G_{\ell, m}$ and are indexed by ℓ -tuples $\alpha = (\alpha_1, \dots, \alpha_\ell)$ of positive integers with $1 \leq \alpha_1 < \dots < \alpha_\ell \leq m$. The Grassmann codes are a special case where $\alpha_i = m - \ell + i$ for $i = 1, \dots, \ell$. It was shown in [6] that the minimum distance of $C_\alpha(\ell, m)$ satisfies the inequality

$$(3) \quad d(C_\alpha(\ell, m)) \leq q^{\delta(\alpha)} \quad \text{where} \quad \delta(\alpha) := \sum_{i=1}^{\ell} (\alpha_i - i).$$

Further, it was conjectured in [6] that the inequality in (3) is, in fact, an equality. We will refer to this conjecture as the Minimum Distance Conjecture, or in short, the MDC. When $\alpha_i = m - \ell + i$ for $i = 1, \dots, \ell$, we have $\delta(\alpha) = \delta$ and so the MDC holds, thanks to (2). In the case $\ell = 2$, the MDC was proved in the affirmative by Chen [3] and, independently, by Guerra and Vincenti [9]. An explicit formula for the length n_α and dimension k_α of $C_\alpha(\ell, m)$ in the case $\ell = 2$ was also given in [3], while [9] gave a general, even if complicated, formula for n_α for arbitrary ℓ . Later, in [8], the MDC was established for Schubert divisors (i.e., in the case $\delta(\alpha) = \delta - 1$) and general formulas for n_α and k_α were obtained, namely,

$$(4) \quad n_\alpha = \sum_{\beta \leq \alpha} q^{\delta(\beta)} \quad \text{and} \quad k_\alpha = \det_{1 \leq i, j \leq \ell} \left(\binom{\alpha_j - j + 1}{i - j + 1} \right),$$

where the above summation is over all ℓ -tuples $\beta = (\beta_1, \dots, \beta_\ell)$ of integers satisfying $1 \leq \beta_1 < \dots < \beta_\ell \leq m$ and $\beta_i \leq \alpha_i$ for $i = 1, \dots, \ell$ and $\delta(\beta) := \sum_{i=1}^{\ell} (\beta_i - i)$. It may be noted the length n_α is asymptotically equal to the conjectured minimum distance $q^{\delta(\alpha)}$ as $q \rightarrow \infty$. An affirmative answer to the MDC was eventually proposed by Xiang [20], where an alternative proof of the inequality in (3) is given and a clever and rather involved proof of the other inequality is also given. While one doesn't doubt the veracity of Xiang's proof, it has been felt that a cleaner and more transparent proof of the MDC would be desirable. This has been expressed to the first named author by some researchers and appears to have been indicated by the MathSciNet review of [20]. With this in view, we give in this paper an alternative

coordinate-free argument to establish the MDC in the general case. Further, we take up the problem of characterizing the minimum weight codewords of Schubert codes and determining their number. In the case of Grassmann codes, there is a nice characterization that was given already by Nogin [14]. To explain this, let us note that the codewords of $C(\ell, m)$ are indexed by elements f of the exterior power $\bigwedge^{m-\ell} V$ and may be denoted by c_f . The map $\bigwedge^{m-\ell} V \rightarrow C(\ell, m)$ given by $f \mapsto c_f$ is a linear bijection. The characterization is simply that c_f is a minimum weight codeword $C(\ell, m)$ if and only if f is decomposable, i.e., $f = f_1 \wedge \dots \wedge f_{m-\ell}$ for some linearly independent $f_1, \dots, f_{m-\ell} \in V$. As the experience with the MDC shows, the case of Schubert codes seems more difficult. In the first place, the map $f \mapsto c_f$ of $\bigwedge^{m-\ell} V \rightarrow C_\alpha(\ell, m)$ is surjective, but not injective, in general. Nonetheless, the codewords of $C_\alpha(\ell, m)$ can still be indexed by $f \in \bigwedge^{m-\ell} V$. It turns out that one needs here a stronger and more subtle notion of decomposability that we call *Schubert decomposability*. We propose a new conjecture that the minimum weight codewords of $C_\alpha(\ell, m)$ are precisely those that correspond to Schubert decomposable elements of $\bigwedge^{m-\ell} V$. We prove several aspects of this conjecture. Thus we show that codewords indexed by Schubert decomposable elements of $\bigwedge^{m-\ell} V$ are minimum weight codewords of $C_\alpha(\ell, m)$ and we also show that if $c_f \in C_\alpha(\ell, m)$ is a minimum weight codeword for some decomposable $f \in \bigwedge^{m-\ell} V$, then f is Schubert decomposable. What remains to be seen is whether every minimum weight codeword of $C_\alpha(\ell, m)$ can be indexed by a decomposable element of $\bigwedge^{m-\ell} V$. We show that this is indeed the case when α is “completely non-consecutive”, i.e., when $\alpha_i - \alpha_{i-1} \geq 2$ for all $i = 1, \dots, \ell$. Thus the new conjecture is established in this case. We also give an explicit lower bound for the number of minimum weight codewords of $C_\alpha(\ell, m)$, and observe that it gives the exact value if our new conjecture is true. Of course in the completely non-consecutive case, this becomes an unconditional result. We also show that unlike Grassmann codes, the minimum weight codewords of a Schubert code do not, in general, generate the code. On the other hand, one knows from the recent work of Piñero [15] that the duals of Schubert codes have the same low minimum distance as that of $C(\ell, m)^\perp$ and moreover the minimum weight codewords of $C_\alpha(\ell, m)^\perp$ generate $C_\alpha(\ell, m)^\perp$.

The results of this paper have a geometric interpretation that may be of independent interest. Indeed, $\Omega_\alpha(\ell, m)$ admits a nondegenerate embedding in \mathbb{P}^{k_α} and using the language of projective systems (see, e.g., [18, §1.1]), we see that determining the minimum distance $d(C_\alpha(\ell, m))$ is equivalent to determining the maximum number of \mathbb{F}_q -rational points in sections of $\Omega_\alpha(\ell, m)$ by hyperplanes in \mathbb{P}^{k_α} since

$$\max\{ |(\Omega_\alpha(\ell, m) \cap H)(\mathbb{F}_q)| : H \text{ hyperplane in } \mathbb{P}^{k_\alpha} \} = n_\alpha - d(C_\alpha(\ell, m)).$$

Furthermore, the number of minimum weight codewords of $C_\alpha(\ell, m)$ is precisely

$$\left| \left\{ H : H \text{ hyperplane in } \mathbb{P}^{k_\alpha}(\mathbb{F}_q) \text{ with } |(\Omega_\alpha(\ell, m) \cap H)(\mathbb{F}_q)| = n_\alpha - d(C_\alpha(\ell, m)) \right\} \right|.$$

2. PRELIMINARIES

In this section, we recall some basic notions and set the notations and terminology used in the rest of this paper. As in the Introduction, a prime power q and integers ℓ, m with $1 \leq \ell \leq m$ will be kept fixed throughout this paper. We have frequently used the notation $A := B$ to mean that A is defined to be equal to B .

2.1. Linear Codes. Let n, k be positive integers. By a $[n, k]_q$ -code we mean a linear k -dimensional subspace of \mathbb{F}_q^n . Let C be a $[n, k]_q$ -code. The parameters n and k are called the *length* and the *dimension* of C , respectively, whereas elements of C are usually referred to as *codewords*. Given a codeword $c = (c_1, \dots, c_n)$ of C , the *Hamming weight* of c will be denoted by $\text{wt}(c)$; this is simply the number of $i \in \{1, \dots, n\}$ for which $c_i \neq 0$. The *minimum distance* of C is denoted by $d(C)$ and can be defined as $\min\{\text{wt}(c) : c \in C \text{ with } c \neq 0\}$. Elements $c \in C$ satisfying $\text{wt}(c) = d(C)$ are called the *minimum weight codewords* of C .

2.2. Grassmann and Schubert Varieties. Let \mathbb{F} be a field (later we will mainly take $\mathbb{F} = \mathbb{F}_q$, but for now it can be an arbitrary field) and V be an m -dimensional vector space over \mathbb{F} . For a nonnegative integer d , we let $\bigwedge^d V$ denote the d th exterior power of V ; this is a vector space over \mathbb{F} of dimension $\binom{m}{d}$. Fixing a basis of V , we can (and will) identify $\bigwedge^m V$ with \mathbb{F} . Also the dual $(\bigwedge^d V)^*$ is canonically identified with $\bigwedge^{m-d} V$. An element f of $\bigwedge^d V$ is said to be *decomposable* if $f \neq 0$ and $f = f_1 \wedge \dots \wedge f_d$ for some $f_1, \dots, f_d \in V$. In general, elements of $\bigwedge^d V$ are \mathbb{F} -linear combinations of decomposable elements. The *annihilator* of any $f \in \bigwedge^d V$ is the subspace of V denoted by V_f and defined by

$$V_f := \{x \in V : f \wedge x = 0\}.$$

Evidently, $f = 0$ if and only if $\dim V_f = m$. Now suppose $d < m$. Then the following characterization is well-known; see, e.g., [12, Thm. 1.1]:

$$(5) \quad f \text{ is decomposable} \iff \dim V_f = d.$$

Note that if f is decomposable and $f = f_1 \wedge \dots \wedge f_d$, then $\{f_1, \dots, f_d\}$ is a basis of V_f . And if $\{g_1, \dots, g_d\}$ is an arbitrary basis of V_f , then $f = \lambda(g_1 \wedge \dots \wedge g_d)$ where $\lambda \in \mathbb{F}$ is a nonzero scalar given by the determinant of the change-of-basis matrix.

As in the Introduction, the Grassmannian $G_{\ell, m} = G_\ell(V)$ may be defined by

$$G_{\ell, m} := \{L : L \text{ is a } \ell\text{-dimensional subspace of } V\}.$$

Elements of $G_{\ell, m}$ can be identified with the points of the projective space $\mathbb{P}(\bigwedge^\ell V)$ via the *Plücker embedding*, which associates to a subspace $L \in G_{\ell, m}$ with \mathbb{F} -basis $\{v_1, \dots, v_\ell\}$ the class $[v_1 \wedge \dots \wedge v_\ell]$ of $v_1 \wedge \dots \wedge v_\ell \in \bigwedge^\ell V$. It is well-known that this is a well-defined embedding under which $G_{\ell, m}$ corresponds to a projective algebraic variety in $\mathbb{P}(\bigwedge^\ell V)$ defined by the vanishing of certain quadratic homogeneous polynomials with integer coefficients. Moreover, the embedding is nondegenerate,

i.e., $G_{\ell,m}$ is not contained in any hyperplane of $\mathbb{P}(\bigwedge^\ell V)$. One can also view $G_{\ell,m}$ as a quotient of $\mathrm{GL}_m(\mathbb{F})$. Indeed, the group $\mathrm{GL}(V)$ of invertible linear maps of $V \rightarrow V$ acts transitively on $G_{\ell,m}$ and so $G_{\ell,m}$ can be viewed as the homogeneous space $\mathrm{GL}(V)/P_\ell$, where P_ℓ is the parabolic subgroup given by the stabilizer of a fixed ℓ -dimensional subspace of V . As this indicates, $G_{\ell,m}$ is a nonsingular variety of dimension $\delta := \ell(m-\ell)$. When $\ell = m$, the Grassmannian is a particularly simple object, namely the singleton set $\{V\}$, or the projective space \mathbb{P}^0 consisting of a single point. Thus, to avoid trivialities, we shall henceforth assume that $1 \leq \ell < m$.

Now let us fix a partial flag $A_1 \subset \cdots \subset A_\ell$ of nonzero subspaces of V . Let $\alpha_i := \dim A_i$ for $i = 1, \dots, \ell$. We may refer to $\alpha := (\alpha_1, \dots, \alpha_\ell)$ as the *dimension sequence* of the partial flag $A_1 \subset \cdots \subset A_\ell$. Note that $1 \leq \alpha_1 < \cdots < \alpha_\ell \leq m$. The *Schubert variety* corresponding to this partial flag depends essentially on the dimension sequence α and is defined by

$$(6) \quad \Omega_\alpha(\ell, m) := \{L \in G_{\ell,m} : \dim(L \cap A_i) \geq i \text{ for all } i = 1, \dots, \ell\}.$$

With respect to the Plücker embedding of $G_{\ell,m}$, the Schubert variety $\Omega_\alpha(\ell, m)$ corresponds to a subset of $\mathbb{P}(\bigwedge^\ell V)$ given by the intersection of $G_{\ell,m}$ with a bunch of Plücker coordinate hyperplanes. As such $\Omega_\alpha(\ell, m)$ is indeed a projective variety that is known to be nondegenerately embedded in $\mathbb{P}^{k_\alpha-1}$, where k_α is as in (4). Note that the elements of $\Omega_\alpha(\ell, m)$ are precisely those $L \in G_{\ell,m}$ for which there is a basis of the form $\{v_1, \dots, v_\ell\}$ with the property $v_i \in A_i$ for $i = 1, \dots, \ell$. Thus

$$\Omega_\alpha(\ell, m) = \{[v_1 \wedge \cdots \wedge v_\ell] : v_1, \dots, v_\ell \in V \text{ linearly independent and } v_i \in A_i \forall i\}.$$

We shall use either of the above two descriptions of $\Omega_\alpha(\ell, m)$ as may be convenient. Moreover, we may reverse the order so as to write $[v_1 \wedge \cdots \wedge v_\ell]$ as $[v_\ell \wedge \cdots \wedge v_1]$.

Now note that α can be divided into consecutive blocks as

$$\alpha = (\alpha_1, \dots, \alpha_{p_1}, \alpha_{p_1+1}, \dots, \alpha_{p_2}, \dots, \alpha_{p_{u-1}+1}, \dots, \alpha_{p_u}, \alpha_{p_u+1}, \dots, \alpha_\ell)$$

so that $1 \leq p_1 < \cdots < p_u < \ell$ and $\alpha_{p_i+1}, \dots, \alpha_{p_{i+1}}$ are consecutive for $0 \leq i \leq u$, where by convention, $p_0 = 0$ and $p_{u+1} = \ell$. If we further require that $\alpha_{p_{i+1}} - \alpha_{p_i} \geq 2$ for $i = 1, \dots, u$, then the nonnegative integer u and the “jump spots” p_1, \dots, p_u are uniquely determined by α . For example, if $\ell = 7$ and $\alpha = (1, 2, 4, 5, 6, 8, 10)$, then $u = 3$ and $(p_1, p_2, p_3) = (2, 5, 6)$. It is an easy consequence of the dimension formula (see, e.g., [8, Lemma 2]) that if the dimension condition in (6) holds at the “jump spots”, then it holds everywhere else; in other words,

$$(7) \quad \Omega_\alpha(\ell, m) = \{L \in G_\ell(A_\ell) : \dim(L \cap A_{p_i}) \geq p_i \text{ for all } i = 1, \dots, u\}.$$

Hereafter, u and $p_0, p_1, \dots, p_u, p_{u+1}$ will denote the unique integers satisfying

$$(8) \quad p_0 := 0 < p_1 < \cdots < p_u < p_{u+1} := \ell, \quad \text{and} \quad \alpha_{p_{i+1}} - \alpha_{p_i} \geq 2 \text{ for } 1 \leq i \leq u$$

and moreover, $\alpha_{p_{i-1}+1}, \dots, \alpha_{p_i}$ are consecutive for $1 \leq i \leq u+1$, that is,

$$(9) \quad \alpha_{p_i-j} = \alpha_{p_i} - j \quad \text{for } 1 \leq i \leq u+1 \text{ and } 1 \leq j < p_i - p_{i-1}.$$

In particular, if α is *completely consecutive*, i.e., if $u = 0$, then (7) shows that $\Omega_\alpha(\ell, m)$ coincides with the Grassmannian $G_\ell(A_\ell)$ of all ℓ -dimensional subspaces of A_ℓ . The other extreme is $u = \ell - 1$, which means $\alpha_{i+1} - \alpha_i \geq 2$ for $i = 1, \dots, \ell - 1$, and we will refer to such α as *completely non-consecutive*. We now define a notion that will play an important role in the sequel.

Definition 2.1. An element f of $\bigwedge^{m-\ell} V$ is said to be *Schubert decomposable* (with respect to the Schubert variety $\Omega_\alpha(\ell, m)$) if f is decomposable, i.e., $f \neq 0$ and $f = f_1 \wedge \dots \wedge f_{m-\ell}$ for some $f_1, \dots, f_{m-\ell} \in V$, and moreover,

$$(10) \quad \dim(V_f \cap A_{p_i}) = a_{p_i} - p_i \quad \text{for all } i = 1, \dots, u.$$

Note that if α is completely consecutive, then $u = 0$ and condition (10) is vacuously true. Thus in this case the notions of decomposable and Schubert decomposable elements are identical. However, in general, a decomposable element need not be Schubert decomposable.

2.3. Grassmann Codes and Schubert Codes. Here, and hereafter, we will assume that the base field \mathbb{F} is the finite field \mathbb{F}_q . Then $G_{\ell, m}$ and $\Omega_\alpha(\ell, m)$ are finite and the number of (\mathbb{F}_q -rational) points in these varieties are n and n_α , which were given explicitly in (1) and (4), respectively. Fix an ordering L_1, \dots, L_{n_α} of the elements of $\Omega_\alpha(\ell, m)$ and representatives P_1, \dots, P_{n_α} in $\bigwedge^\ell V$ such that each P_i is a decomposable element of the form $v_\ell \wedge \dots \wedge v_1$ with $v_i \in A_i$ for all $i = 1, \dots, \ell$ and $L_j = V_{P_j}$ for $j = 1, \dots, n_\alpha$. The choice we make here of ordering v_i in the descending order in i is merely a matter of convenience and will be found suitable when we use induction on ℓ . Needless to say, the element $v_\ell \wedge \dots \wedge v_1$ differs only in sign from $v_1 \wedge \dots \wedge v_\ell$. At any rate, we have a natural evaluation map

$$(11) \quad \bigwedge^{m-\ell} V \rightarrow \mathbb{F}_q^{n_\alpha} \quad \text{defined by} \quad f \mapsto c_f, \quad \text{where} \quad c_f := (f \wedge P_1, \dots, f \wedge P_{n_\alpha}).$$

The *Schubert code* $C_\alpha(\ell, m)$ is defined as the image of this evaluation map. The Grassmann code $C(\ell, m)$ is a special case when $\alpha_i = m - \ell + i$ for $i = 1, \dots, \ell$. Note that a different choice of representatives results in a code that is monomially equivalent to $C_\alpha(\ell, m)$. With this in view, given any $f \in \bigwedge^{m-\ell} V$, instead of $f \wedge P_i$ we may often write $f \wedge L_i$ or $f(L_i)$. This is an abuse of notation, but perfectly unambiguous when we are only interested in the vanishing or nonvanishing of the scalar $f(L_i)$. This, for instance, is the case in the definition of the *support* of f :

$$(12) \quad W(f) := \{L \in \Omega_\alpha(\ell, m) : f(L) \neq 0\}.$$

It is clear that the cardinality of $W(f)$ is $\text{wt}(c_f)$, i.e., the Hamming weight of the codeword c_f of $C_\alpha(\ell, m)$. In particular,

$$(13) \quad d(C_\alpha(\ell, m)) = \min\{|W(f)| : f \in \bigwedge^{m-\ell} V \text{ and } W(f) \text{ is nonempty}\},$$

where for any finite set S , we let $|S|$ denote the cardinality of S .

We note that (11) gives a surjective map of $\bigwedge^{m-\ell} V$ onto $C_\alpha(\ell, m)$, but this map is, in general, not injective. In fact, its kernel is of dimension $\binom{m}{\ell} - k_\alpha$, where k_α is as in (1). Moreover, from an alternative expression for k_α given in [8, eq. (4)], it is readily seen that $k_\alpha < \binom{m}{\ell}$ if and only if $\Omega_\alpha(\ell, m) \neq G_{\ell, m}$. Note also that

$$(14) \quad f \wedge L = 0 \text{ if } f \in \bigwedge^{m-\ell} V \text{ and } L \in \Omega_\alpha(\ell, m) \text{ are such that } V_f \cap L \neq \{0\}.$$

This easily verifiable observation may be used tacitly in the sequel.

3. MINIMUM DISTANCE OF SCHUBERT CODES

For the remainder of this paper, fix a prime power q , positive integers ℓ, m with $\ell < m$ and an m -dimensional vector space V over \mathbb{F}_q . Also let us fix a partial flag $A_1 \subset \cdots \subset A_\ell$ of nonzero subspaces of V and let $\alpha = (\alpha_1, \dots, \alpha_\ell)$ be its dimension sequence. For any integer j , we set $A_j := \{0\}$ if $j \leq 0$ and $A_j := V$ if $j > \ell$, by convention. Given any $v_1, \dots, v_r \in V$, we shall denote by $\langle v_1, \dots, v_r \rangle$ the linear subspace of V generated by v_1, \dots, v_r . Likewise if L' is a subspace of V and $v \in V$, then by $\langle L', v \rangle$ we denote the subspace of V generated by v and the elements of L' . Given a finite dimensional vector space W , a subspace of W of dimension $\dim W - 1$ may be referred to as a *hyperplane* in W . We shall use the notation and terminology introduced in the previous section. In particular, given any $f \in \bigwedge^{m-\ell} V$, we denote by c_f the corresponding codeword in the Schubert code $C_\alpha(\ell, m)$.

In case $\ell > 1$, we will denote by α' the $(\ell - 1)$ -tuple $(\alpha_1, \dots, \alpha_{\ell-1})$, which is the dimension sequence of the truncated partial flag $A_1 \subset \cdots \subset A_{\ell-1}$. Moreover, when $\ell > 1$ and $f \in \bigwedge^{m-\ell} V$ is given, we put

$$E := \{x \in A_\ell : c_{f \wedge x} \text{ is the zero codeword in } C_{\alpha'}(\ell - 1, m)\} \quad \text{and} \quad F := A_\ell \setminus E.$$

It is clear that E is a subspace of A_ℓ . Naturally, E and F depend on f and to make this dependence explicit, we may sometimes denote them by E_f and F_f . However, in most situations there will be a fixed $f \in \bigwedge^{m-\ell} V$, and we will drop the subscript so as to simply write E and F . The following lemma is a simple, but crucial, observation made by Xiang [20]. We include a proof for the sake of completeness.

Lemma 3.1. *Assume that $\ell > 1$ and $f \in \bigwedge^{m-\ell} V$ is given. Let E be as above and let t be a nonnegative integer such that $\text{codim}_{A_\ell} E \leq t$. Then $A_{\ell-t} \subseteq E$.*

Proof. If $t = 0$ or $t \geq \ell$, then the result holds trivially. Thus, assume that $1 \leq t < \ell$. Suppose, on the contrary, there is some $x \in A_{\ell-t} \setminus E$. Then there are $x_i \in A_i$ for $1 \leq i \leq \ell - 1$ such that

$$(15) \quad f \wedge x \wedge x_{\ell-1} \wedge \cdots \wedge x_1 \neq 0.$$

In particular, $x, x_{\ell-t}, \dots, x_{\ell-1}$ are linearly independent. Now if y is any nonzero element of $\langle x, x_{\ell-t}, \dots, x_{\ell-1} \rangle$, then we can replace x or some x_j ($\ell - t \leq j \leq \ell - 1$) by y to obtain a basis of $\langle x, x_{\ell-t}, \dots, x_{\ell-1} \rangle$ consisting of y and all except one among

$x, x_{\ell-t}, \dots, x_{\ell-1}$. So it follows from (15) that $f \wedge y \wedge y_{\ell-1} \wedge \dots \wedge y_1 \neq 0$ for some $y_i \in A_i$ ($1 \leq i \leq \ell-1$). Consequently, $y \notin E$. Thus $E \cap \langle x, x_{\ell-t}, \dots, x_{\ell-1} \rangle = \{0\}$. Hence $\dim E \leq \alpha_\ell - t - 1$, i.e., $\text{codim}_{A_\ell} E \geq t + 1$, which is a contradiction. \square

The next lemma is also due to Xiang [20]. We give a coordinate-free proof.

Lemma 3.2. *Assume that $\ell > 1$ and $f \in \bigwedge^{m-\ell} V$ is given. Let α' and E be as defined above. Also let*

$$Z(\alpha', f) = \{(L', x) \in \Omega_{\alpha'}(\ell-1, m) \times A_\ell : (f \wedge x)(L') \neq 0\}$$

and let $\phi : Z(\alpha', f) \rightarrow W(f)$ be the map given by $(L', x) \mapsto \langle L', x \rangle$. Then ϕ is well-defined and surjective. Moreover, given any $L \in W(f)$, the following holds.

- (i) If $L \not\subseteq A_{\ell-1}$ then $|\phi^{-1}(L)| = q^{\ell-1}(q-1)$.
- (ii) If $L \subseteq A_{\ell-1}$ and if $t := \text{codim}_{A_\ell} E$, then $|\phi^{-1}(L)| \leq q^{\ell-1}(q^t-1)$.

Proof. It is clear that ϕ is well-defined (i.e., $\langle L', x \rangle \in W(f)$ whenever (L', x) is in $Z(\alpha', f)$) and that ϕ is surjective. Now let $L \in W(f)$.

(i) Suppose $L \not\subseteq A_{\ell-1}$. Since $L \in \Omega_\alpha(\ell, m)$, we see that $\dim(L \cap A_{\ell-1}) = \ell-1$. Now if $(L', x) \in \phi^{-1}(L)$, then L' is a $(\ell-1)$ -dimensional subspace of $L \cap A_{\ell-1}$ and hence $L' = L \cap A_{\ell-1}$. On the other hand, x can be an arbitrary element of $L \setminus L'$. Thus $|\phi^{-1}(L)| = q^\ell - q^{\ell-1} = q^{\ell-1}(q-1)$.

(ii) Suppose $L \subseteq A_{\ell-1}$ and $t := \text{codim}_{A_\ell} E$. Let $(L', x) \in \phi^{-1}(L)$. Observe that

$$(16) \quad L' \cap A_{\ell-t} = L \cap A_{\ell-t}.$$

Indeed, the inclusion \subseteq is obvious, whereas if there exists $u \in (L \cap A_{\ell-t}) \setminus L'$, then $L = \langle L', u \rangle$ and by Lemma 3.1, $u \in E$, which implies that $(f \wedge u)(L') = 0$ and hence $f(L) = 0$, which contradicts the assumption that $L \in W(f)$.

Now let $L_t := L \cap A_{\ell-t}$. From (16), we see that L' is necessarily a hyperplane in L containing L_t . The number of such hyperplanes is equal to the number of hyperplanes in L/L_t . Thus if $r := \dim L_t$ and $N' := (q^{\ell-r}-1)/(q-1)$, then we see that there are at most N' choices for L' . Note that since $L \in \Omega_\alpha(\ell, m)$, we have $r \geq \ell-t$ and hence $N' \leq (q^t-1)/(q-1)$. Moreover, x has to be in $L \setminus L'$ and so there are at most $q^\ell - q^{\ell-1}$ possibilities for x . It follows that $|\phi^{-1}(L)| \leq q^{\ell-1}(q^t-1)$. \square

Lemma 3.3. *Assume that $\ell > 1$. Let $f \in \bigwedge^{m-\ell} V$ be such that $c_f \neq 0$ and let E and F be the corresponding sets as defined above. Also let $t := \text{codim}_{A_\ell} E$. Then*

$$(17) \quad \text{wt}(c_f) \geq \frac{1}{q^{\ell-1}(q-1)} \sum_{x \in F \cap A_{\ell-1}} \text{wt}(c_{f \wedge x}) + \frac{1}{q^{\ell-1}(q^t-1)} \sum_{x \in F \setminus A_{\ell-1}} \text{wt}(c_{f \wedge x}).$$

Moreover, the inequality above is strict if the inequality in part (ii) of Lemma 3.2 is strict for some $L \in W(f)$ with $L \subseteq A_{\ell-1}$.

Proof. Let α' , $Z(\alpha', f)$, and $\phi : Z(\alpha', f) \rightarrow W(f)$ be as in Lemma 3.2. Then

$$|Z(\alpha', f)| = \sum_{L \in W(f)} |\phi^{-1}(L)| = \sum_{\substack{L \in W(f) \\ L \not\subseteq A_{\ell-1}}} |\phi^{-1}(L)| + \sum_{\substack{L \in W(f) \\ L \subseteq A_{\ell-1}}} |\phi^{-1}(L)|.$$

On the other hand, considering the fibres of the projection $Z(\alpha', f) \rightarrow F$, we obtain

$$|Z(\alpha', f)| = \sum_{x \in F} \text{wt}(c_{f \wedge x}) = \sum_{x \in F \cap A_{\ell-1}} \text{wt}(c_{f \wedge x}) + \sum_{x \in F \setminus A_{\ell-1}} \text{wt}(c_{f \wedge x}).$$

Now observe that for any $L' \in \Omega_{\alpha'}(\ell-1, m)$ and $x \in A_{\ell}$, we have

$$L' + \langle x \rangle \in W(f) \text{ and } L' + \langle x \rangle \not\subseteq A_{\ell-1} \iff x \in F \setminus A_{\ell-1} \text{ and } L' \in W(f \wedge x).$$

This implies the first equality below, which in turn, yields the second inequality.

$$\sum_{\substack{L \in W(f) \\ L \not\subseteq A_{\ell-1}}} |\phi^{-1}(L)| = \sum_{x \in F \cap A_{\ell-1}} \text{wt}(c_{f \wedge x}) \text{ and } \sum_{\substack{L \in W(f) \\ L \subseteq A_{\ell-1}}} |\phi^{-1}(L)| = \sum_{x \in F \setminus A_{\ell-1}} \text{wt}(c_{f \wedge x}).$$

Hence if we let $\theta_1 := |\{L \in W(f) : L \not\subseteq A_{\ell-1}\}|$ and $\theta_2 := |\{L \in W(f) : L \subseteq A_{\ell-1}\}|$, then $\text{wt}(c_f) = \theta_1 + \theta_2$ and from Lemma 3.2, we see that

$$\sum_{x \in F \cap A_{\ell-1}} \text{wt}(c_{f \wedge x}) = \theta_1 q^{\ell-1}(q-1) \text{ and } \sum_{x \in F \setminus A_{\ell-1}} \text{wt}(c_{f \wedge x}) \leq \theta_2 q^{\ell-1}(q^t - 1).$$

This implies (17). Moreover, if $|\phi^{-1}(L)| < q^{\ell-1}(q^t - 1)$ for some $L \in W(f)$ with $L \subseteq A_{\ell-1}$, then it is clear that the inequality in (17) is strict. \square

For ease of reference, we state the following result for which a short proof is given in [6, Prop. 5.2], while an alternative proof is given in [20, Thm. 1]. Yet another proof will be sketched in Remark 5.4.

Proposition 3.4. $d(C_{\alpha}(\ell, m)) \leq q^{\delta(\alpha)}.$

We are now ready to show that the MDC holds in the affirmative, in general. We shall also see that the proof also gives us some information about the minimum weight codewords of $C_{\alpha}(\ell, m)$.

Theorem 3.5. $d(C_{\alpha}(\ell, m)) = q^{\delta(\alpha)}$. Moreover, if $\ell > 1$ and $f \in \bigwedge^{m-\ell} V$ is such that c_f is a minimum weight codeword in $C_{\alpha}(\ell, m)$, then $c_{f \wedge x}$ is a minimum weight codeword in $C_{\alpha'}(\ell-1, m)$ for every $x \in F$ and furthermore, we must have either (i) $t = 1$ and $t' = 0$, or (ii) $t' = t \geq 2$. $\alpha_{\ell} - \alpha_{\ell-1} = 1$, and equality holds in (17). Here α' , E , F and t are as in Lemma 3.2 and $t' := \text{codim}_{A_{\ell-1}}(E \cap A_{\ell-1})$.

Proof. In view of Proposition 3.4, in order to show that $d(C_{\alpha}(\ell, m)) = q^{\delta(\alpha)}$, it suffices to show that

$$(18) \quad \text{wt}(c_f) \geq q^{\delta(\alpha)} \quad \text{for every } f \in \bigwedge^{m-\ell} V \text{ such that } c_f \neq 0.$$

We now proceed to prove (18) by induction on ℓ ($1 \leq \ell < m$). The initial case can be deduced from facts about Grassmann codes or first order projective Reed-Muller codes, but we will give a direct and self-contained proof. The induction step will make use of the above lemma.

First, suppose $\ell = 1$ and $f \in \bigwedge^{m-\ell} V$. Then f is necessarily decomposable, say $f = f_1 \wedge \cdots \wedge f_{m-1}$ for some $f_1, \dots, f_{m-1} \in V$. Further, suppose $c_f \neq 0$. Then we must have $A_1 \not\subseteq V_f = \langle f_1, \dots, f_{m-1} \rangle$. Consequently, $A_1 + V_f = V$ and hence $\dim A_1 \cap V_f = \alpha_1 - 1$. Noting that $W(f) = \{\langle x \rangle : x \in A_1 \setminus (A_1 \cap V_f)\}$, we obtain

$$\text{wt}(c_f) = |W(f)| = \frac{q^{\alpha_1} - q^{\alpha_1-1}}{q-1} = q^{\alpha_1-1} = q^{\delta(\alpha)}.$$

Next, suppose $1 < \ell < m$ and (18) holds for positive values of ℓ smaller than the given one. Note that $t \geq 1$ since $c_f \neq 0$. Note also that $t' := \text{codim}_{A_{\ell-1}}(E \cap A_{\ell-1})$ satisfies $t' \leq t$. Indeed, if s elements of E extend a basis of $E \cap A_{\ell-1}$ to a basis of E , then no nontrivial linear combination of these s elements is in $A_{\ell-1}$ and hence $s \leq \alpha_\ell - \alpha_{\ell-1}$, which implies $t' \leq t$. We shall now divide the proof into two cases.

Case 1. $t = 1$.

In this case, by Lemma 3.1, $A_{\ell-1} \subseteq E$. Hence $F \setminus A_{\ell-1} = F$ and $F \cap A_{\ell-1} = \emptyset$. It follows that $|F \setminus A_{\ell-1}| = |F| = |A_\ell| - |E| = q^{\alpha_\ell} - q^{\alpha_{\ell-1}}$. Moreover, by the induction hypothesis, $\text{wt}(c_{f \wedge x}) \geq q^{\delta(\alpha')}$, for every $x \in F$. Now (17) reduces to

$$\text{wt}(c_f) \geq \frac{1}{q^{\ell-1}(q-1)} \sum_{x \in F} \text{wt}(c_{f \wedge x}) \geq \frac{1}{q^{\ell-1}(q-1)} (q^{\alpha_\ell} - q^{\alpha_{\ell-1}}) q^{\delta(\alpha')} = q^{\delta(\alpha)}.$$

Thus (18) is proved in this case. Also, it is clear that if $\text{wt}(c_{f \wedge x}) > q^{\delta(\alpha')}$ for some $x \in F$, then $\text{wt}(c_f) > q^{\delta(\alpha)}$. Note also that $t' = 0$ in this case, since $A_{\ell-1} \subseteq E$.

Case 2. $t \geq 2$.

First note that, with t and t' as above,

$$|F \setminus A_{\ell-1}| = (|A_\ell| - |E|) - (|A_{\ell-1}| - |E \cap A_{\ell-1}|) = (q^{\alpha_\ell} - q^{\alpha_{\ell-1}}) - (q^{\alpha_{\ell-1}} - q^{\alpha_{\ell-1}-t'}).$$

We will now consider three different subcases as follows.

Subcase 2.1. $\alpha_\ell - \alpha_{\ell-1} \geq 2$.

Here, using the expression for $|F \setminus A_{\ell-1}|$ obtained above, we see that

$$|F \setminus A_{\ell-1}| - (q^{\alpha_\ell} - q^{\alpha_{\ell-1}}) = (q^{\alpha_{\ell-1}} - q^{\alpha_{\ell-1}-t} - q^{\alpha_{\ell-1}}) + q^{\alpha_{\ell-1}-t'} > 0,$$

where the last inequality follows by noting that $q^{\alpha_{\ell-1}} \geq 2q^{\alpha_{\ell-1}-2} \geq q^{\alpha_{\ell-1}-t} + q^{\alpha_{\ell-1}-1}$, since $t \geq 2$ and $\alpha_\ell - \alpha_{\ell-1} \geq 2$. Hence by (17) and the induction hypothesis,

$$\text{wt}(c_f) \geq \frac{1}{q^{\ell-1}(q-1)} |F \setminus A_{\ell-1}| q^{\delta(\alpha')} > \frac{1}{q^{\ell-1}(q-1)} (q^{\alpha_\ell} - q^{\alpha_{\ell-1}}) q^{\delta(\alpha')} = q^{\delta(\alpha)}.$$

Thus we obtain (18) with, in fact, a strict inequality.

Subcase 2.2. $\alpha_\ell - \alpha_{\ell-1} = 1$ and $t' \neq t$.

Here, $t' \leq t - 1$ and so using the expression for $|F \setminus A_{\ell-1}|$ obtained earlier, we see that $|F \setminus A_{\ell-1}| - (q^{\alpha_\ell} - q^{\alpha_\ell-1}) = (q^{\alpha_\ell-1-t'} - q^{\alpha_\ell-t}) \geq 0$, and also that strict inequality holds when $t' = 0$. Hence by (17) and the induction hypothesis,

$$\text{wt}(c_f) > \frac{1}{q^{\ell-1}(q-1)}(q^{\alpha_\ell} - q^{\alpha_\ell-1})q^{\delta(\alpha')} = q^{\delta(\alpha)},$$

where the above inequality is strict either because $t' > 0$ in which case the second summation in (17) is nonempty and contributes a positive term or because $t' = 0$ in which case $|F \setminus A_{\ell-1}| > (q^{\alpha_\ell} - q^{\alpha_\ell-1})$. This yields (18), with a strict inequality.

Subcase 2.3. $\alpha_\ell - \alpha_{\ell-1} = 1$ and $t' = t$.

In this subcase of Case 2, we readily see that

$$|F \setminus A_{\ell-1}| = q^{\alpha_\ell-t-1}(q^t - 1)(q - 1) \quad \text{and} \quad |F \cap A_{\ell-1}| = q^{\alpha_\ell-t-1}(q^t - 1).$$

Hence from (17) and the induction hypothesis, we obtain

$$\text{wt}(c_f) \geq \frac{q^{\alpha_\ell-t-1}(q^t - 1)}{q^{\ell-1}}q^{\delta(\alpha')} + \frac{q^{\alpha_\ell-t-1}}{q^{\ell-1}}q^{\delta(\alpha')} = q^{\delta(\alpha)}.$$

Also, it is clear that if $\text{wt}(c_{f \wedge x}) > q^{\delta(\alpha')}$ for some $x \in F$ or if the inequality in (17) is strict, then $\text{wt}(c_f) > q^{\delta(\alpha)}$.

Thus (18) is proved in all cases. Hence by induction on ℓ we conclude that $d(C_\alpha(\ell, m)) = q^{\delta(\alpha)}$. The remaining assertions in the statement of the theorem are also clear from the proof. \square

4. ANNIHILATORS OF DECOMPOSABLE ELEMENTS

Now that we know the minimum distance of Schubert codes, it is natural to ask for a classification as well as enumeration of the minimum weight codewords. In this section, we shall take some preliminary steps towards such a classification by analysing the intersections of annihilators of decomposable elements of $\bigwedge^{m-\ell} V$ with the constituent subspaces of the partial flag defining the given Schubert variety.

As in Section 3, integers ℓ, m with $1 \leq \ell < m$, an m -dimensional vector space V over \mathbb{F}_q , a partial flag $A_1 \subset \dots \subset A_\ell$ of nonzero subspaces of V with dimension sequence $\alpha = (\alpha_1, \dots, \alpha_\ell)$ will be kept fixed throughout this section. Also, recall (from §2.2) that for any $f \in \bigwedge^{m-\ell} V$, by V_f we denote the annihilator of f .

Lemma 4.1. *Suppose $f \in \bigwedge^{m-\ell} V$ is a decomposable element with $c_f \neq 0$. Then*

$$\alpha_i - \ell \leq \dim(V_f \cap A_i) \leq \alpha_i - i \quad \text{for all } i = 1, \dots, \ell.$$

In particular, $\dim(V_f \cap A_\ell) = \alpha_\ell - \ell$.

Proof. Since f is decomposable, $\dim V_f = m - \ell$. Hence for $1 \leq i \leq \ell$,

$$m \geq \dim(V_f + A_i) = \dim(V_f) + \dim(A_i) - \dim(V_f \cap A_i) = m - \ell + \alpha_i - \dim(V_f \cap A_i),$$

and thus $\dim(V_f \cap A_i) \geq \alpha_i - \ell$. Further, since $c_f \neq 0$, there are $x_j \in A_j$ for $j = 1, \dots, \ell$ such that $f \wedge x_\ell \wedge \dots \wedge x_1 \neq 0$. Consequently, for each $i = 1, \dots, \ell$,

there are linearly independent $x_1, \dots, x_i \in A_i$ such that $f \wedge x_i \wedge \dots \wedge x_1 \neq 0$ and therefore $V_f \cap \langle x_1, \dots, x_i \rangle = \{0\}$. This implies that $\dim(V_f \cap A_i) \leq \alpha_i - i$. \square

It turns out that the attainment of the upper bound given in Lemma 4.1 for $\dim(V_f \cap A_i)$ has a nice characterization when $i = \ell - 1$.

Lemma 4.2. *Assume that $\ell > 1$. Let $f \in \bigwedge^{m-\ell} V$ be a decomposable element with $c_f \neq 0$, and E the corresponding subspace as in §3. Then:*

$$\dim(V_f \cap A_{\ell-1}) = \alpha_{\ell-1} - (\ell - 1) \iff \text{codim}_{A_\ell} E = 1.$$

Proof. Since $c_f \neq 0$, there are $x_i \in A_i$ for $1 \leq i \leq \ell$ such that $f \wedge (x_\ell \wedge \dots \wedge x_1) \neq 0$. Consequently, $V_f \cap \langle x_1, \dots, x_\ell \rangle = \{0\}$ and $x_\ell \notin E$. In particular, $E \neq A_\ell$.

Suppose $\dim(V_f \cap A_{\ell-1}) = \alpha_{\ell-1} - (\ell - 1)$. Observe that $x_1, \dots, x_{\ell-1} \in E$. Indeed, if $x_i \notin E$ for some $i \leq \ell - 1$, then there exist $y_j \in A_j$ for $j = 1, \dots, \ell - 1$ such that $(f \wedge x_i) \wedge (y_{\ell-1} \wedge \dots \wedge y_1) \neq 0$; consequently, $\langle x_i, y_1, \dots, y_{\ell-1} \rangle$ is a subspace of $A_{\ell-1}$ of dimension ℓ such that $V_f \cap \langle x_i, y_1, \dots, y_{\ell-1} \rangle = \{0\}$, and this yields a contradiction. It follows that $(V_f \cap A_\ell) + \langle x_1, \dots, x_{\ell-1} \rangle$ is an $(\alpha_\ell - 1)$ -dimensional subspace of E . Since $E \neq A_\ell$, we conclude that $\text{codim}_{A_\ell} E = 1$.

Conversely, suppose $\text{codim}_{A_\ell} E = 1$. Note that $\dim(V_f \cap A_{\ell-1}) \leq \alpha_{\ell-1} - (\ell - 1)$, thanks to Lemma 4.1. In case $\dim(V_f \cap A_{\ell-1}) < \alpha_{\ell-1} - (\ell - 1)$, there exists $y_\ell \in A_{\ell-1}$ such that $y_\ell \notin (V_f \cap A_{\ell-1}) + \langle x_1, \dots, x_{\ell-1} \rangle$. Clearly, $f \wedge (y_\ell \wedge x_{\ell-1} \wedge \dots \wedge x_1) \neq 0$. It follows that $x_{\ell-1}$ and y_ℓ are linearly independent elements of $A_{\ell-1}$ and neither of them is in E . Changing $x_{\ell-1} \wedge y_\ell$ to $x_{\ell-1} \wedge z$ or $z \wedge y_\ell$ for any nonzero $z \in \langle x_{\ell-1}, y_\ell \rangle$, we see that $\langle x_{\ell-1}, y_\ell \rangle \cap E = \{0\}$, and so $\text{codim}_{A_\ell} E > 1$, which is a contradiction. \square

5. SCHUBERT DECOMPOSABILITY AND MINIMUM WEIGHT CODEWORDS

We will continue to use the notation and terminology of the previous two sections. For the dimension sequence $\alpha = (\alpha_1, \dots, \alpha_\ell)$ of the fixed partial flag $A_1 \subset \dots \subset A_\ell$, we let u and $p_0, p_1, \dots, p_u, p_{u+1}$ denote the unique integers satisfying (8) and (9). Recall that $f \in \bigwedge^{m-\ell} V$ is said to be Schubert decomposable (w.r.t. $\Omega_\alpha(\ell, m)$) if $\dim(V_f \cap A_{p_i}) = \alpha_{p_i} - p_i$ for $i = 1, \dots, u$. Note that this equality for dimension also holds for $i = u + 1$, thanks to Lemma 4.1.

We shall now proceed to relate Schubert decomposability with the minimum weight codewords of $C_\alpha(\ell, m)$. We begin with a simple and basic observation.

Lemma 5.1. *Let $f \in \bigwedge^{m-\ell} V$ be Schubert decomposable. Then c_f is nonzero.*

Proof. Since $\dim(V_f \cap A_{p_1}) = \alpha_{p_1} - p_1$, by extending a basis of $V_f \cap A_{p_1}$ to A_{p_1} , we can find a p_1 -dimensional subspace L_1 of A_{p_1} such that $L_1 \cap V_f = \{0\}$. Now since $\alpha_1, \dots, \alpha_{p_1}$ are consecutive, we see that

$$\dim(L_1 \cap A_{p_1-j}) \geq \dim(L_1 \cap A_{p_1-j+1}) - 1 \quad \text{for each } 1 \leq j < p_1$$

and this implies that $\dim(L_1 \cap A_i) \geq i$ for $i = 1, \dots, p_1$. Hence we can choose $x_i \in A_i$ for $i = 1, \dots, p_1$ such that $\{x_1, \dots, x_{p_1}\}$ is a basis of L_1 . Next, observe that $\dim(V_f \cap A_{p_2}) + L_1 = \alpha_{p_2} - p_2 + p_1$. Now since $\alpha_{p_1+1}, \dots, \alpha_{p_2}$ are consecutive, by arguing as before, we can find $x_i \in A_i$ for $i = p_1 + 1, \dots, p_2$ such that $\{x_1, \dots, x_{p_2}\}$ forms a basis of a p_2 -dimensional subspace L_2 of A_{p_2} such that $L_2 \cap V_f = \{0\}$ and $L_1 \subset L_2$. Continuing in this manner, we obtain linearly independent $x_1, \dots, x_\ell \in V$ such that $x_i \in A_i$ for $i = 1, \dots, \ell$ and $V_f \cap \langle x_1, \dots, x_\ell \rangle = \{0\}$. Consequently, $L := [x_1 \wedge \dots \wedge x_\ell]$ is in $\Omega_\alpha(\ell, m)$ and since f is decomposable, we must have $f(L) \neq 0$. Thus $c_f \neq 0$. \square

Our next result is a refined version of Lemma 3.2 with an additional hypothesis of Schubert decomposability.

Lemma 5.2. *Assume that $\ell > 1$ and $f \in \bigwedge^{m-\ell} V$ is Schubert decomposable. Let $\alpha', E, Z(\alpha', f)$ and $\phi : Z(\alpha', f) \rightarrow W(f)$ be as in Lemma 3.2, and $t := \text{codim}_{A_\ell} E$. Then $t = 1$ or $t = \ell - p_u$. Moreover,*

$$(19) \quad |\phi^{-1}(L)| = q^{\ell-1}(q^t - 1) \quad \text{for every } L \in W(f).$$

Proof. If $t = 1$, then (19) follows from part (i) of Lemma 3.2. Now suppose $t > 1$. Then from Lemma 4.1 and 4.2, we see that $\dim(V_f \cap A_{\ell-1}) = \alpha_{\ell-1} - \ell$. In particular, $p_u < \ell - 1$, since f is Schubert decomposable. Moreover, since $\alpha_{p_u+1}, \dots, \alpha_{\ell-1}$ are consecutive, if for some j with $p_u + 1 \leq j \leq \ell - 1$, we had $\dim(V_f \cap A_j) \geq \alpha_j - j$, then we would obtain

$$\dim(V_f \cap A_{\ell-1}) \geq \alpha_j - j = \alpha_{\ell-1} - (\ell - 1 - j) - j = \alpha_{\ell-1} - (\ell - 1),$$

which is a contradiction. Thus, for each $j = p_u + 1, \dots, \ell - 1$, we have

$$(20) \quad \dim(V_f \cap A_j) \leq \alpha_j - j, \text{ and in particular, } \dim(V_f \cap A_{p_u+1}) \leq \alpha_{p_u+1} - p_u - 2.$$

Now since $\dim(V_f \cap A_{p_i}) = \alpha_{p_i} - p_i$ for $i = 1, \dots, u$, by arguing as in the proof of Lemma 5.1, we obtain linearly independent elements $x_1, \dots, x_{p_u} \in V$ such that $V_f \cap \langle x_1, \dots, x_{p_u} \rangle = \{0\}$ and $\langle x_1, \dots, x_{p_i} \rangle \subseteq A_{p_i}$ for each $i = 1, \dots, u$. Hence the sum $(V_f \cap A_{p_u+1}) + \langle x_1, \dots, x_{p_u} \rangle$ is a subspace of A_{p_u+1} (as well as $A_{\ell-1}$) of dimension $\leq \alpha_{p_u+1} - 2$, and so we can find y_1, y_2 in A_{p_u+1} and $y_3, \dots, y_{\ell-p_u}$ in $A_{\ell-1}$ such that $x_1, \dots, x_{p_u}, y_1, y_2, \dots, y_{\ell-p_u}$ are linearly independent and moreover, $V_f \cap \langle x_1, \dots, x_{p_u}, y_1, y_2, \dots, y_{\ell-p_u} \rangle = \{0\}$. Since f is decomposable, it follows that $f \wedge x_1 \wedge \dots \wedge x_{p_u} \wedge y_1 \wedge \dots \wedge y_{\ell-p_u} \neq 0$. Consequently, $y_1 \in A_{p_u+1} \setminus E$ and so $A_{p_u+1} \not\subseteq E$. Hence from Lemma 3.1, we see that $\text{codim}_{A_\ell} E > \ell - (p_u + 1)$, i.e., $\text{codim}_{A_\ell} E \geq \ell - p_u$. We will now proceed to show that $\text{codim}_{A_\ell} E = \ell - p_u$. To this end, observe that $A_{p_u} \subseteq E$. Indeed, if there exists $x \in A_{p_u} \setminus E$, then $f \wedge x \wedge z_1 \wedge \dots \wedge z_{\ell-1} \neq 0$ for some $z_i \in A_i$ ($1 \leq i \leq \ell - 1$). In particular, $f \wedge x \wedge z_1 \wedge \dots \wedge z_{p_u} \neq 0$, and hence $V_f \cap \langle x, z_1, \dots, z_{p_u} \rangle = \{0\}$. This implies that

$\dim(V_f \cap A_{p_u}) \leq \alpha_{p_u} - p_u - 1$, which contradicts the assumption that f is Schubert decomposable. Thus, $A_{p_u} \subseteq E$ and hence $(V_f \cap A_\ell) + A_{p_u} \subseteq E$. Consequently,

$$(\alpha_\ell - \ell) + \alpha_{p_u} - (\alpha_{p_u} - p_u) \leq \dim E, \quad \text{that is,} \quad \text{codim}_{A_\ell} E \leq \ell - p_u.$$

Thus we have proved that $t = \ell - p_u$.

Now fix any $L \in W(f)$. Let $L_t := L \cap A_{\ell-t} = L \cap A_{p_u}$. Observe that $\dim L_t \geq p_u$, since $L \in \Omega_\alpha(\ell, m)$. Also, since $L \cap V_f = \{0\}$, we see that $(V_f \cap A_{p_u}) \cap L_t = \{0\}$. Hence the Schubert decomposability of f implies that $\alpha_{p_u} - p_u + \dim L_t \leq \alpha_{p_u}$, i.e., $\dim L_t \leq p_u$. Thus $\dim L_t = p_u$. Next, we claim that

$$(L', x) \in \phi^{-1}(L) \iff L' \text{ is a hyperplane in } L \text{ containing } L_t \text{ and } x \in L \setminus L'.$$

The implication \Rightarrow is clear since we have seen in the proof of Lemma 3.2 that if $(L', x) \in \phi^{-1}(L)$, then $L' \cap A_{\ell-t} = L_t$. For the other implication, we use the fact that $\ell - t = p_u$ to deduce that $L' \cap A_{p_i} = L \cap A_{p_i}$ for $i = 1, \dots, u$. Hence in view of (7), we see that $L' \in \Omega_{\alpha'}(\ell - 1, m)$ for every hyperplane L' in $L \in \Omega_\alpha(\ell, m)$. Further, since $x \in L \setminus L'$, we conclude $(L', x) \in \phi^{-1}(L)$. Thus the claim is proved. As a consequence, we see that $|\phi^{-1}(L)| = N'(q^\ell - q^{\ell-1})$, where N' is exactly the number of hyperplanes in L containing L_t . Now $N' = (q^{\ell-(\ell-t)} - 1)/(q - 1)$, exactly as in the proof of Lemma 3.2. This yields the desired formula for $|\phi^{-1}(L)|$. \square

Theorem 5.3. *If $f \in \bigwedge^{m-\ell} V$ is Schubert decomposable, then c_f is a minimum weight codeword of $C_\alpha(\ell, m)$.*

Proof. We use induction on ℓ . If $\ell = 1$, then the desired result follows from Lemma 5.1 since every nonzero codeword of $C_\alpha(1, m)$ is a minimum weight codeword. Now suppose $\ell > 1$ and that the result holds for values of ℓ smaller than the given one. Let $f \in \bigwedge^{m-\ell} V$ be Schubert decomposable, and let E and F be the corresponding subsets of A_ℓ as in Section 3. Given any $x \in F$, we note that $g := f \wedge x$ is a decomposable element of $\bigwedge^{m-(\ell-1)} V$ satisfying $V_f \subset V_g$ and $c_g \neq 0$. In particular, for any i with $1 \leq i \leq u$, we have $V_f \cap A_{p_i} \subseteq V_g \cap A_{p_i}$. On the other hand, $\dim V_g \cap A_{p_i} \leq \alpha_{p_i} - p_i = \dim V_f \cap A_{p_i}$, thanks to Lemma 4.1 and the Schubert decomposability of f . It follows that g is Schubert decomposable and hence by the induction hypothesis, c_g is a minimum weight codeword of $C_{\alpha'}(\ell - 1, m)$. Now, proceeding as in the proof of Lemma 3.3, except for applying Lemma 5.2 in place of part (ii) of Lemma 3.2, we see that

$$\sum_{x \in F \cap A_{\ell-1}} \text{wt}(c_{f \wedge x}) = \theta_1 q^{\ell-1} (q - 1) \quad \text{and} \quad \sum_{x \in F \setminus A_{\ell-1}} \text{wt}(c_{f \wedge x}) = \theta_2 q^{\ell-1} (q^t - 1),$$

where $\theta_1 := |\{L \in W(f) : L \not\subseteq A_{\ell-1}\}|$ and $\theta_2 := |\{L \in W(f) : L \subseteq A_{\ell-1}\}|$. Since $\text{wt}(c_f) = \theta_1 + \theta_2$ and since $f \wedge x$ is of weight $q^{\delta(\alpha')}$ for every $x \in F$, we obtain

$$(21) \quad \text{wt}(c_f) = \frac{1}{q^{\ell-1}(q-1)} |F \setminus A_{\ell-1}| q^{\delta(\alpha')} + \frac{1}{q^{\ell-1}(q^t-1)} |F \cap A_{\ell-1}| q^{\delta(\alpha')}.$$

In case $t = 1$, this gives

$$\text{wt}(c_f) = \frac{1}{q^{\ell-1}(q-1)} |F| q^{\delta(\alpha')} = \frac{1}{q^{\ell-1}(q-1)} (q^{\alpha_\ell} - q^{\alpha_{\ell-1}}) q^{\delta(\alpha')} = q^{\delta(\alpha)}.$$

Now suppose $t > 1$. Then $t = \ell - p_u$ by Lemma 5.2 and so $p_u < \ell$. Consequently, $\alpha_\ell - \alpha_{\ell-1} = 1$. Thus, $A_{\ell-1}$ is a hyperplane in A_ℓ , and therefore

$$\dim E \cap A_{\ell-1} \geq \dim E - 1.$$

Moreover, in view of Lemmas 4.2 and 4.1, we see that

$$\dim(V_f \cap A_{\ell-1}) = \alpha_{\ell-1} - \ell = \alpha_\ell - \ell - 1 = (\dim V_f \cap A_\ell) - 1.$$

Hence we can find some $z \in V_f \cap A_\ell$ such that $z \notin V_f \cap A_{\ell-1}$. Since $V_f \subseteq E$, we see that $z \in E \setminus (E \cap A_{\ell-1})$ and therefore

$$\dim E \cap A_{\ell-1} \leq \dim E - 1.$$

It follows that $\dim E \cap A_{\ell-1} = \dim E - 1$, and hence $t' := \text{codim}_{A_{\ell-1}}(E \cap A_{\ell-1})$ is equal to t . Consequently, as in Subcase 2.3 of the proof of Theorem 3.5,

$$|F \setminus A_{\ell-1}| = q^{\alpha_\ell - t - 1} (q^t - 1)(q - 1) \quad \text{and} \quad |F \cap A_{\ell-1}| = q^{\alpha_\ell - t - 1} (q^t - 1).$$

Using this together with (21), we obtain $\text{wt}(c_f) = q^{\delta(\alpha)}$. Since Theorem 3.5 shows that $q^{\delta(\alpha)}$ is the minimum distance of $C_\alpha(\ell, m)$, the proof is complete. \square

Remark 5.4. Except for the last line in the proof of above theorem, the fact that $q^{\delta(\alpha)}$ is the minimum distance of $C_\alpha(\ell, m)$ has not been used anywhere. In fact, our proof of Theorem 5.3 shows that if $f \in \bigwedge^{m-\ell} V$ is Schubert decomposable, then c_f is a codeword of $C_\alpha(\ell, m)$ of weight $q^{\delta(\alpha)}$. Since it is easy to construct $(m - \ell)$ -dimensional subspaces W of V such that $\dim W \cap A_{p_i} = \alpha_{p_i} - p_i$ for $i = 1, \dots, u+1$ (see, e.g., the proof of Theorem 7.5; equations (27) and (30) here give, in fact, an explicit formula for the cardinality of the set Λ_α of such subspaces), we can deduce that $d(C_\alpha(\ell, m)) \leq q^{\delta(\alpha)}$. This provides an alternative proof of Proposition 3.4.

We will now prove that the converse of Theorem 5.3 is true provided that the element f of $\bigwedge^{m-\ell} V$ is assumed to be decomposable.

Theorem 5.5. *Assume that $f \in \bigwedge^{m-\ell} V$ is decomposable. If c_f is a minimum weight codeword of $C_\alpha(\ell, m)$, then f is Schubert decomposable.*

Proof. We use induction on ℓ . If $\ell = 1$, then $u = 0$ and there is nothing to prove. Suppose $\ell > 1$ and that the result holds for values of ℓ smaller than the given one. Let $f \in \bigwedge^{m-\ell} V$ be a decomposable element such that c_f is a minimum weight codeword of $C_\alpha(\ell, m)$. In particular, $c_f \neq 0$ and if E and F denote the subsets of A_ℓ associated to f , as in Section 3. then $t := \text{codim}_{A_\ell} E$ satisfies $1 \leq t \leq \alpha_\ell$. Moreover, by Theorem 3.5, $c_{f \wedge x}$ is a minimum weight codeword of $C_{\alpha'}(\ell - 1, m)$ for every $x \in F$ and furthermore, we either have $t = 1$ or we have $t > 1$ and $\alpha_\ell - \alpha_{\ell-1} = 1$.

Note that for an arbitrary $x \in F$, by induction hypothesis we see that $g := f \wedge x$ is Schubert decomposable. Hence $\dim V_g \cap A_{p_i} = \alpha_{p_i} - p_i$ for all $i = 1, \dots, u$. Now since f is decomposable, so is g ; moreover, V_f is a hyperplane in V_g . Hence

$$(22) \quad q_i := \dim V_f \cap A_{p_i} = \alpha_{p_i} - p'_i \quad \text{where} \quad p'_i = p_i \text{ or } p_i + 1 \quad \text{for } i = 1, \dots, u.$$

Let us also note that it suffices to show that $V_g \cap A_{p_u} \subseteq V_f \cap A_{p_u}$ because in that case $V_f \cap A_{p_u} = V_g \cap A_{p_u}$, the other inclusion being trivial. Consequently, $\dim V_f \cap A_{p_i} = \alpha_{p_i} - p_i$ for all $i = 1, \dots, u$. Thus f is Schubert decomposable. We will now divide the proof into two cases according as $t = 1$ or $t > 1$.

Case 1. $t = 1$

In this case $p_u \leq \ell - 1$ and by Lemma 3.1, $A_{p_u} \subseteq E$. Let x be an arbitrary element of F and as before, let $g = f \wedge x$. Let $y \in V_g \cap A_{p_u}$. Since $y \in V_g = V_f + \langle x \rangle$, we can write $y = z + \lambda x$ for some $z \in V_f$ and $\lambda \in \mathbb{F}_q$. Also since $y \in A_{p_u}$, we find $y \in E$ and so $c_{f \wedge y} = 0$. Moreover, $c_{f \wedge z} = 0$ simply because $z \in V_f$. It follows that

$$0 = c_{f \wedge y} = c_{f \wedge z} + \lambda c_{f \wedge x} = \lambda c_{f \wedge x} \quad \text{and hence} \quad \lambda = 0 \quad \text{so that} \quad y \in V_f \cap A_{p_u}.$$

Thus $V_g \cap A_{p_u} \subseteq V_f \cap A_{p_u}$ and so, as noted before, f is Schubert decomposable.

Case 2. $t > 1$

In this case $\alpha_\ell - \alpha_{\ell-1} = 1$ and so $p_u < \ell - 1$, i.e., $p_u + 2 \leq \ell$. Recall that as per our convention $p_{u+1} := \ell$ and so in view of (22) and Lemma 4.1, we set $p'_{u+1} := \ell$ and $q_{u+1} := \alpha_\ell - \ell$. Now we can recursively find $y_1, \dots, y_{m-\ell}$ such that

$$V_f \cap A_{p_i} = \langle y_1, \dots, y_{q_i} \rangle \quad \text{for } i = 1, \dots, u+1 \quad \text{and} \quad V_f = \langle y_1, \dots, y_{m-\ell} \rangle.$$

In view of the dimension formulas (22) (that are also valid for $i = u+1$), this ensures that no nontrivial linear combination of $y_{q_i+1}, \dots, y_{m-\ell}$ is in A_{p_i} for each $i = 1, \dots, u+1$. By recursively extending these bases of $V_f \cap A_{p_i}$ to A_{p_i} , we can also find $z_1, \dots, z_\ell \in A_\ell$ such that

$$A_{p_i} = \langle y_1, \dots, y_{q_i}, z_1, \dots, z_{p'_i} \rangle \quad \text{for } i = 1, \dots, u+1.$$

Now consider the subspaces L and L' of A_ℓ defined by

$$L = \langle z_1, \dots, z_\ell \rangle \quad \text{and} \quad L' = \langle z_1, \dots, z_{p'_u}, z_{p'_u+2}, \dots, z_\ell \rangle.$$

It is clear that $\dim L = \ell = \dim L' + 1$ and also that

$$\dim(L \cap A_{p_i}) = \dim(L' \cap A_{p_i}) = \dim \langle z_1, \dots, z_{p'_i} \rangle = p'_i \geq p_i \quad \text{for } i = 1, \dots, u.$$

Hence in view of 7, we see that $L \in \Omega_\alpha(\ell, m)$ and $L' \in \Omega_{\alpha'}(\ell - 1, m)$. Moreover by our choice of z_1, \dots, z_ℓ , it is clear that $V_f \cap L = \{0\}$. Since f is decomposable, it follows from (14) that $c_f(L) \neq 0$. Hence if we let $x := z_{p'_u+1}$, then we find $c_{f \wedge x}(L') \neq 0$. It follows that $x \in F$ and so induction hypothesis applies to $g = f \wedge x$

for this choice of x . Thus, g is Schubert decomposable. Moreover, if $y \in V_g \cap A_{p_u}$, then being an element of $V_g = V_f + \langle x \rangle$, we can write

$$y = z + \lambda x \text{ for some } z \in \langle y_1, \dots, y_{m-\ell} \rangle \text{ and } \lambda \in \mathbb{F}_q.$$

On the other hand, being an element of A_{p_u} , we see that

$$y \in \langle y_1, \dots, y_{q_u}, z_1, \dots, z_{p'_u} \rangle.$$

Thus if $\lambda \neq 0$, then $x = z_{p'_u+1}$ can be expressed as a linear combination of $y_1, \dots, y_{m-\ell}, z_1, \dots, z_{p'_u}$, which contradicts the choice of y 's and z 's. So $\lambda = 0$ and $y \in V_f \cap A_{p_u}$. Thus $V_g \cap A_{p_u} \subseteq V_f \cap A_{p_u}$ and so f is Schubert decomposable. \square

In view of Theorems 5.3 and 5.5, we make the following conjecture.

Conjecture 5.6. *Minimum weight codewords of the Schubert code $C_\alpha(\ell, m)$ are precisely the codewords corresponding to Schubert decomposable elements of $\bigwedge^{m-\ell} V$.*

6. COMPLETELY NON-CONSECUTIVE CASE

We will continue to use the notation and terminology of the last three sections. The main result of this section is an affirmative answer to Conjecture 5.6 when the dimension sequence $\alpha = (\alpha_1, \dots, \alpha_\ell)$ of the fixed partial flag $A_1 \subset \dots \subset A_\ell$ is completely non-consecutive, i.e., when $\alpha_i - \alpha_{i-1} \geq 2$ for $1 < i \leq \ell$.

Theorem 6.1. *Assume that α is completely non-consecutive. If c is a minimum weight codeword of $C_\alpha(\ell, m)$, then $c = c_h$ for some decomposable $h \in \bigwedge^{m-\ell} V$.*

Proof. We will use induction on ℓ . The result clearly holds when $\ell = 1$ since every nonzero element of $\bigwedge^{m-1} V$ is decomposable. Suppose $\ell > 1$ and the result is true for values of ℓ smaller than the given one. Let c be a minimum weight codeword of $C_\alpha(\ell, m)$. Fix $f \in \bigwedge^{m-\ell} V$ such that $c = c_f$, and let E, F be as in Section 3. Since $\alpha_\ell - \alpha_{\ell-1} \geq 2$, by Theorem 3.5 we see that $\text{codim}_{A_\ell} E = 1$ and $f \wedge x$ is a minimum weight codeword of $C_{\alpha'}(\ell-1, m)$ for every $x \in F$. Moreover, $A_{\ell-1} \subseteq E$, thanks to Lemma 3.2. Thus we can and will choose a basis $\{e_1, \dots, e_m\}$ of V such that

$$(23) \quad A_i = \langle e_1, \dots, e_{\alpha_i} \rangle \text{ for } i = 1, \dots, \ell \text{ and } E = \langle e_1, \dots, e_{\alpha_{\ell-1}} \rangle.$$

Let $x := e_{\alpha_\ell}$. Clearly, $x \in F$ and hence $f \wedge x$ is a minimum weight codeword of $C_{\alpha'}(\ell-1, m)$. Moreover, α' is completely non-consecutive. So by induction hypothesis, $c_{f \wedge x} = c_g$ for some decomposable $g \in \bigwedge^{m-\ell+1} V$. Moreover, by Theorem 5.5, g is Schubert decomposable, and so $\dim V_g \cap A_i = \alpha_i - i$ for $i = 1, \dots, \ell-1$. Thus we can recursively choose $z_1, \dots, z_{\ell-1}$ such that

$$z_1 \in A_1 \setminus (V_g \cap A_1) \text{ and } z_i \in A_i \setminus (\langle z_1, \dots, z_{i-1} \rangle + V_g \cap A_i) \text{ for } i = 2, \dots, \ell-1.$$

In particular, $z_1, \dots, z_{\ell-1}$ span an $(\ell-1)$ -dimensional subspace, say $B_{\ell-1}$ of $A_{\ell-1}$ such that $A_{\ell-1} = B_{\ell-1} + (V_g \cap A_{\ell-1})$. This implies that $V_g \cap B_{\ell-1} = \{0\}$. Also since $\dim V_g = m - \ell + 1$ and $x \notin A_{\ell-1}$, we see that $\dim V_g \cap (B_{\ell-1} + \langle x \rangle) \geq 1$.

Hence V_g contains an element of the form $b + x$ for some $b \in B_{\ell-1}$. Consequently, we can find $g_1, \dots, g_{m-\ell} \in V$ such that $g_1, \dots, g_{\alpha_{\ell-1}-(\ell-1)}$ span $V_g \cap A_{\ell-1}$ and

$$g = g_1 \wedge \dots \wedge g_{m-\ell} \wedge (b + x) = g' \wedge b + g' \wedge x, \quad \text{where } g' := g_1 \wedge \dots \wedge g_{m-\ell}.$$

Note that $V_g \cap A_{\ell-1} = \langle g_1, \dots, g_{\alpha_{\ell-1}-(\ell-1)} \rangle \subseteq V_{g'} \cap A_{\ell-1} \subseteq V_g \cap A_{\ell-1}$. Thus

$$V_{g'} \cap A_{\ell-1} = V_g \cap A_{\ell-1} \quad \text{and} \quad \dim(V_{g'} \cap A_{\ell-1}) = \alpha_{\ell-1} - (\ell - 1).$$

We claim that $c_{g' \wedge b} = 0$. This is clear if $b \in V_{g'}$. Now suppose $b \notin V_{g'}$. Then

$$V_{g' \wedge b} \cap A_{\ell-1} = (V_{g'} + \langle b \rangle) \cap A_{\ell-1} = (V_{g'} \cap A_{\ell-1}) + \langle b \rangle \text{ has dimension } \alpha_{\ell-1} - \ell + 2$$

and therefore $V_{g' \wedge b} \cap A_{\ell-1}$ has nonzero intersection with any $(\ell - 1)$ -dimensional subspace of $A_{\ell-1}$. In particular, $U \cap V_{g' \wedge b} \neq \{0\}$ for every $U \in \Omega_{\alpha'}(\ell - 1, m)$. Thus, in view of (14), the claim is proved. From the claim, it follows that $c_g = c_{g' \wedge x}$. Writing each of $g_1, \dots, g_{m-\ell}$ as a linear combination of e_1, \dots, e_m and noting that $x = e_{\alpha_\ell}$, we see that $g' \wedge x = h \wedge x$ where h is a decomposable element of $\bigwedge^{m-\ell} V$ of the form $h_1 \wedge \dots \wedge h_{m-\ell}$, where each of $h_1, \dots, h_{m-\ell}$ is in the $(m - 1)$ -dimensional space V spanned by $\{e_1, \dots, e_m\} \setminus \{x\}$. We will now proceed to prove that $c_f = c_h$.

Let $L \in \Omega_\alpha(\ell, m)$ and let $P = u_\ell \wedge \dots \wedge u_1$ with $u_i \in A_i$ for $1 \leq i \leq \ell$, be the representative of L in $\bigwedge^\ell V$ among the fixed representatives P_1, \dots, P_{n_α} as in §2.3. We wish to show that $c_f(P) := f \wedge u_\ell \wedge \dots \wedge u_1$ is equal to $c_h(P) := h \wedge u_\ell \wedge \dots \wedge u_1$. Since $c_{f \wedge x} = c_{h \wedge x}$, we readily see that $f \wedge x \wedge u_{\ell-1} \wedge \dots \wedge u_1 = h \wedge x \wedge u_{\ell-1} \wedge \dots \wedge u_1$. We will now consider two cases. First, suppose $u_\ell \in E$. Then $c_{f \wedge u_\ell}$ is the zero codeword in $C_{\alpha'}(\ell - 1, m)$ and hence $c_f(P) = 0$. On the other hand, by (23) and our choice of h , we see that $V_h + E$ is a subspace of V' . Since $\dim V_h + \dim L = m > \dim V'$, we must have $V_h \cap L \neq \{0\}$ and so by (14), we obtain $c_h(L) = 0$ as well. Not suppose $u_\ell \notin E$. Then $u_\ell = v_\ell + \lambda x$ for a unique $v_\ell \in E$ and $\lambda \in \mathbb{F}_q$ with $\lambda \neq 0$. As in the previous case, $f \wedge v_\ell \wedge u_{\ell-1} \wedge \dots \wedge u_1 = 0 = h \wedge v_\ell \wedge u_{\ell-1} \wedge \dots \wedge u_1$. Consequently,

$$c_h(P) = \lambda (h \wedge x \wedge u_{\ell-1} \wedge \dots \wedge u_1) = \lambda (f \wedge x \wedge u_{\ell-1} \wedge \dots \wedge u_1) = f \wedge u_\ell \wedge u_{\ell-1} \wedge \dots \wedge u_1,$$

and thus $c_h(P) = c_f(P)$. This establishes $c_f = c_h$ and so the theorem is proved. \square

As an immediate consequence of the above theorem, we see that Conjecture 5.6 holds in the affirmative when α is completely non-consecutive.

Corollary 6.2. *Assume that α is completely non-consecutive. Then the minimum weight codewords of $C_\alpha(\ell, m)$ are precisely the codewords corresponding to Schubert decomposable elements of $\bigwedge^{m-\ell} V$. More precisely, for any $c \in C_\alpha(\ell, m)$,*

$$c \text{ has minimum weight} \iff c = c_h \text{ for some Schubert decomposable } h \in \bigwedge^{m-\ell} V.$$

Proof. Follows from Theorems 5.3, 5.5 and 6.1. \square

7. ENUMERATION AND GENERATION

In this section, we consider the problem of enumerating the number of minimum weight codewords of Schubert codes and also of determining whether or not the minimum weight codewords generate the Schubert code $C_\alpha(\ell, m)$. The case of Grassmann codes, which is when $\alpha_i = m - \ell + i$ for $i = 1, \dots, \ell$, is well-known. Here we know that the number of minimum weight codewords is $(q-1)\binom{m}{\ell}_q$ and also that the minimum weight codewords of the Grassmann code $C(\ell, m)$ generate $C(\ell, m)$. Both these assertions follow readily from Nogin's characterization of the minimum weight codewords as those that correspond to decomposable elements of $\bigwedge^{m-\ell} V$. In the case of Schubert codes, we have noted earlier that the map given by $f \mapsto c_f$ from $\bigwedge^{m-\ell} V$ onto $C_\alpha(\ell, m)$ is "many-to-one". But for studying the minimum weight codewords of $C_\alpha(\ell, m)$, it may suffice to consider the restriction of this map to the set of Schubert decomposable elements of $\bigwedge^{m-\ell} V$, and examine to what extent it is injective. This is done in the next two lemmas.

Lemma 7.1. *Let $f, g \in \bigwedge^{m-\ell} V$ be Schubert decomposable elements such that $c_f = c_g$. Then $V_f \cap A_{p_i} = V_g \cap A_{p_i}$ for all $i = 1, \dots, u+1$.*

Proof. Assume the contrary, i.e, suppose $V_f \cap A_{p_i} \neq V_g \cap A_{p_i}$ for some $i \leq u+1$. We will assume that i is the least positive integer with this property. Then

$$(24) \quad V_f \cap A_{p_j} = V_g \cap A_{p_j} \quad \text{for } 0 \leq j < i, \quad \text{where } p_0 = 0 \text{ and } A_0 := \{0\}.$$

Since $\dim(V_f \cap A_{p_i}) = \dim(V_g \cap A_{p_i})$, we see that $V_f \cap A_{p_i} \not\subseteq V_g \cap A_{p_i}$. So there is some $x \in V_f \cap A_{p_i}$ such that $x \notin V_g \cap A_{p_i}$. By (24), $x \notin A_{p_j}$ for $0 \leq j < i$. Since f, g are Schubert decomposable, we can recursively choose $x_1, \dots, x_\ell \in A_\ell$ such that

$$(V_g \cap A_{p_j}) + \langle x_1, \dots, x_{p_j} \rangle = A_{p_j} \quad \text{for } j = 1, \dots, u+1 \quad \text{and} \quad x_{p_i} = x,$$

where $p_{u+1} = \ell$. Now let $L = \langle x_1, \dots, x_\ell \rangle$. By our choice of x_1, \dots, x_ℓ , it is clear that from (7) that $L \in \Omega_\alpha(\ell, m)$ and $V_g \cap L = \{0\}$. Since g is decomposable, this implies $c_g(L) \neq 0$. On the other hand, since $x \in V_f \cap L$, we see from (14) that $c_f(L) = 0$. Thus, $c_f \neq c_g$, which contradicts the hypothesis. \square

A partial converse of the above result is also true.

Lemma 7.2. *Let $f, g \in \bigwedge^{m-\ell} V$ be Schubert decomposable elements such that $V_f \cap A_{p_i} = V_g \cap A_{p_i}$ for all $i = 1, \dots, u+1$. Then $c_f = \lambda c_g$ for some $\lambda \in \mathbb{F}_q \setminus \{0\}$.*

Proof. Let $r_i := \alpha_{p_i} - p_i$ for $1 \leq i \leq u+1$ and $r := r_{u+1}$. We can recursively find linearly independent $f_1, \dots, f_r, e_1, \dots, e_\ell \in A_\ell$ such that for each $i = 1, \dots, u+1$,

$$V_f \cap A_{p_i} = V_g \cap A_{p_i} = \langle f_1, \dots, f_{r_i} \rangle \quad \text{and} \quad A_{p_i} = (V_f \cap A_{p_i}) + \langle e_1, \dots, e_{p_i} \rangle.$$

Extend $\{f_1, \dots, f_r\}$ to bases $\{f_1, \dots, f_{m-\ell}\}$ and $\{f_1, \dots, f_r, g_1, \dots, g_{m-\alpha_\ell}\}$ of V_f and V_g respectively, such that $f = f_1 \wedge \dots \wedge f_{m-\ell}$ and $g = f_1 \wedge \dots \wedge f_r \wedge g_1 \wedge \dots \wedge g_{m-\alpha_\ell}$. Note that $V_f \cap \langle e_1, \dots, e_\ell \rangle = \{0\}$ and thus $\{f_1, \dots, f_{m-\ell}, e_1, \dots, e_\ell\}$ is a basis

of V . In particular, for each $i = 1, \dots, m - \alpha_\ell$, we can write $g_i = x_i + y_i + z_i$ for unique $x_i \in \langle f_1, \dots, f_r \rangle$, $y_i \in \langle f_{r+1}, \dots, f_{m-\ell} \rangle$ and $z_i \in \langle e_1, \dots, e_\ell \rangle$. Hence, by multilinearity, we see that g is a finite sum of elements of the form

$$h = f_1 \wedge \dots \wedge f_r \wedge h_1 \wedge \dots \wedge h_{m-\alpha_\ell}, \text{ where } h_i \in \{x_i, y_i, z_i\} \text{ for } i = 1, \dots, m - \alpha_\ell.$$

Now if $h_i = x_i$ for some i , then clearly $h = 0$. Also, if $h_i = z_i$ for some i , then we find that h is a decomposable element of $\bigwedge^{m-\ell} V$ such that

$$\dim(V_h \cap A_\ell) \geq \dim\langle f_1, \dots, f_r, z_i \rangle = \alpha_\ell - \ell + 1$$

and thus $c_h = 0$, because otherwise Lemma 4.1 is contradicted. It follows that $c_g = c_{h^*}$, where $h^* := f_1 \wedge \dots \wedge f_r \wedge y_1 \wedge \dots \wedge y_{m-\alpha_\ell}$. In view of Theorem 5.3, we see that $c_g \neq 0$ and hence $h^* \neq 0$. Consequently, $y_1, \dots, y_{m-\alpha_\ell}$ are linearly independent elements of $\langle f_{r+1}, \dots, f_{m-\ell} \rangle$, and therefore $y_1 \wedge \dots \wedge y_{m-\alpha_\ell}$ and $f_{r+1} \wedge \dots \wedge f_{m-\ell}$ differ by a nonzero scalar. This implies that $c_f = \lambda c_g$ for some $\lambda \in \mathbb{F}_q \setminus \{0\}$. \square

Remark 7.3. It may be noted that with hypothesis as in Lemma 7.2, the stronger conclusion that $f = \lambda g$ for some $\lambda \in \mathbb{F}_q \setminus \{0\}$ or equivalently, $V_f = V_g$, is not true, in general. Indeed this is indicated by the proof and examples are easy to construct. For instance, if $\ell = 2$, $m = 4$, and $\alpha = (2, 4)$, then $f = e_1 \wedge e_3$ and $g = e_1 \wedge (e_2 + e_3)$ are Schubert decomposable elements of $\bigwedge^2 V$ such that $c_f = c_g$, but f and g do not differ by a scalar. Here e_1, e_2, e_3, e_4 denote the elements of a fixed basis of V .

The following lemma is a variant of [8, Lem. 3], but with a simpler formula and a more direct proof.

Lemma 7.4. *Let B be a finite-dimensional vector space over \mathbb{F}_q and let A be a subspace of B and R a subspace of A . Suppose $b = \dim B$, $a = \dim A$ and $r = \dim R$. Let u be any integer with $0 \leq u \leq b$ and, as before, let $G_u(B)$ denote the Grassmannian of u -dimensional subspaces of B . Then*

$$|\{U \in G_u(B) : U \cap A = R\}| = \begin{bmatrix} b-a \\ u-r \end{bmatrix}_q q^{(a-r)(u-r)}.$$

Proof. Let $\mathfrak{U} := \{U \in G_u(B) : U \cap A = R\}$. We have a natural surjective map

$$\psi : \mathfrak{U} \rightarrow G_{u-r}(B/A) \text{ given by } U \mapsto \frac{U+A}{A} \simeq \frac{U}{U \cap A} = \frac{U}{R}.$$

Note that an arbitrary element of $G_{u-r}(B/A)$ is of the form T/A , where T is a subspace of B containing A with $\dim T = a + u - r$. Fix such T/A . Then

$$\psi^{-1}(T/A) = \{U \in G_u(B) : U \cap A = R \text{ and } A + U = T\}$$

To estimate the cardinality of this fiber, let us fix an ordered basis $\{x_1, \dots, x_a\}$ of A such that $\{x_1, \dots, x_r\}$ is a basis of R . Now T has an ordered basis of the form $\{x_1, \dots, x_a, y_1, \dots, y_{u-r}\}$, and $U := \langle x_1, \dots, x_r, y_1, \dots, y_{u-r} \rangle$ is in $\psi^{-1}(T/A)$. Moreover every element of $\psi^{-1}(T/A)$ can be obtained in this manner by choosing $z_1, \dots, z_{u-r} \in T$ such that $\langle z_1, \dots, z_{u-r} \rangle \cap A = \{0\}$ and z_1, \dots, z_{u-r} are linearly

independent. Since $|T| = q^{a+u-r}$ and $|A| = q^a$, the number of ordered $(u-r)$ -tuples (z_1, \dots, z_{u-r}) with this property is

$$(25) \quad (q^{a+u-r} - q^a) (q^{a+u-r} - q^{a+1}) \dots (q^{a+u-r} - q^{a+u-r-1})$$

Two ordered $(u-r)$ -tuples (y_1, \dots, y_{u-r}) and (z_1, \dots, z_{u-r}) give rise to the same subspace if and only if

$$\begin{bmatrix} z_1 \\ \vdots \\ z_{u-r} \end{bmatrix} = \begin{bmatrix} & \vdots & \\ & \vdots & \\ P & \vdots & Q \\ & \vdots & \\ & \vdots & \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_r \\ y_1 \\ \vdots \\ y_{u-r} \end{bmatrix}$$

for some $(u-r) \times r$ matrix P and $(u-r) \times (u-r)$ nonsingular matrix Q with entries in \mathbb{F}_q . Indeed, in that case the two ordered bases $\{x_1, \dots, x_r, y_1, \dots, y_{u-r}\}$ and $\{x_1, \dots, x_r, z_1, \dots, z_{u-r}\}$ will differ by a nonsingular $u \times u$ matrix of the form

$$\begin{bmatrix} I_r & \vdots & \mathbf{0} \\ \dots & \dots & \dots \\ P & \vdots & Q \end{bmatrix}$$

where I_r denotes the identity matrix of size $r \times r$ and $\mathbf{0}$ denotes the $r \times (u-r)$ matrix all of whose entries are zero. The number of ways in which matrices P and Q of the kind above can be chosen is clearly given by

$$(26) \quad q^{r(u-r)} (q^{u-r} - 1) (q^{u-r} - q) \dots (q^{u-r} - q^{u-r-1}).$$

It follows that the cardinality of $\psi^{-1}(T/A)$ is obtained by dividing the expression in (25) by that in (26). Thus

$$|\psi^{-1}(T/A)| = q^{(a-r)(u-r)} \quad \text{and hence} \quad |\mathfrak{U}| = \begin{bmatrix} b-a \\ u-r \end{bmatrix}_q q^{(a-r)(u-r)}.$$

□

We are now ready to prove the main result of this section.

Theorem 7.5. *The number of codewords of $C_\alpha(\ell, m)$ corresponding to Schubert decomposable elements of $\bigwedge^{m-\ell} V$ is equal to*

$$N_\alpha := (q-1)q^P \prod_{j=0}^u \begin{bmatrix} \alpha_{p_{j+1}} - \alpha_{p_j} \\ p_{j+1} - p_j \end{bmatrix}_q$$

where, as per our usual conventions, $p_0 = 0$, $p_{u+1} = \ell$, and $\alpha_0 = 0$, and where

$$P = \sum_{j=1}^u p_j (\alpha_{p_{j+1}} - \alpha_{p_j} - p_{j+1} + p_j).$$

Consequently, the number of minimum weight codewords of $C_\alpha(\ell, m)$ is at least N_α . Moreover, if α is completely non-consecutive, then the number of minimum weight codewords of $C_\alpha(\ell, m)$ is exactly N_α .

Proof. Let us temporarily denote by S_α the set that we wish to enumerate, i.e., let

$$S_\alpha := \{c_f : f \in \bigwedge^{m-\ell} V \text{ is Schubert decomposable}\}.$$

Note that by Theorem 5.3, elements of S_α are minimum weight codewords of $C_\alpha(\ell, m)$ and in particular, nonzero elements of $\mathbb{F}_q^{n_\alpha}$. Consider the relation \sim on S_α defined by $c \sim c' \Leftrightarrow c = \lambda c'$ for some $\lambda \in \mathbb{F}_q \setminus \{0\}$. Clearly, \sim is an equivalence relation and each equivalence class has cardinality $(q-1)$. Thus if we denote by \mathfrak{S}_α the set of all equivalence classes, then $|S_\alpha| = (q-1)|\mathfrak{S}_\alpha|$. On the other hand, there is a similar equivalence relation (viz., proportionality) on the set of all decomposable elements of $\bigwedge^{m-\ell} V$, and the map $f \mapsto V_f$ sending a decomposable element to its annihilator clearly gives a bijection between the set of equivalence classes and the Grassmannian $G_{m-\ell}(V)$ of $(m-\ell)$ -dimensional subspaces of V . This equivalence relation preserves Schubert decomposability and the set of equivalence classes of Schubert decomposable elements of $\bigwedge^{m-\ell} V$ is clearly in bijection with

$$\Lambda_\alpha := \{W \in G_{m-\ell}(V) : \dim W \cap A_{p_i} = \alpha_{p_i} - p_i \text{ for all } i = 1, \dots, u+1\}.$$

For any $c \in S_\alpha$, denote by $[c]$ its equivalence class in \mathfrak{S}_α . Then the map

$$\theta : \Lambda_\alpha \rightarrow \mathfrak{S}_\alpha \quad \text{given by} \quad \langle w_1, \dots, w_{m-\ell} \rangle \mapsto [c_{w_1 \wedge \dots \wedge w_{m-\ell}}]$$

is clearly well-defined and surjective. By Lemmas 7.1 and 7.2, for any $W, W' \in \Lambda_\alpha$,

$$\theta(W) = \theta(W') \Leftrightarrow W \cap A_{p_i} = W' \cap A_{p_i} \text{ for all } i = 1, \dots, u+1 \Leftrightarrow W \cap A_\ell = W' \cap A_\ell.$$

It follows that for any $[c] \in \mathfrak{S}_\alpha$, the fibre $\theta^{-1}([c])$ is in bijection with the set of all $W \in G_{m-\ell}(V)$ such that $W \cap A_\ell$ is equal to a fixed $(\alpha_\ell - \ell)$ -dimensional subspace, say W_ℓ , of A_ℓ . Hence, by Lemma 7.4,

$$(27) \quad |\Lambda_\alpha| = \sum_{[c] \in \mathfrak{S}_\alpha} |\theta^{-1}([c])| = \begin{bmatrix} m - \alpha_\ell \\ m - \ell - (\alpha_\ell - \ell) \end{bmatrix}_q q^{\ell(m-\alpha_\ell)} |\mathfrak{S}_\alpha| = q^{\ell(m-\alpha_\ell)} |\mathfrak{S}_\alpha|.$$

Now let $r_i := \alpha_{p_i} - p_i$ for $1 \leq i \leq u+1$ and consider the following sequence of maps

$$\Lambda_\alpha \xrightarrow{\pi_{u+1}} \Lambda_{u+1} \xrightarrow{\pi_u} \Lambda_u \xrightarrow{\pi_{u-1}} \dots \xrightarrow{\pi_2} \Lambda_2 \xrightarrow{\pi_1} \Lambda_1$$

where for $1 \leq j \leq u+1$, the set Λ_j is defined by

$$\Lambda_j := \{U \in G_{r_j}(A_{p_j}) : \dim U \cap A_{p_i} = \alpha_{p_i} - p_i \text{ for } 1 \leq i < j\},$$

while for $1 \leq j \leq u+1$, the map $\pi_j : \Lambda_{j+1} \rightarrow \Lambda_j$ is defined by $\pi_j(U) = U \cap A_{p_j}$ for any $U \in \Lambda_{j+1}$, where, by convention, we have set

$$\Lambda_{u+2} := \Lambda_\alpha, \quad r_{u+2} := m - \ell, \quad A_{p_{u+2}} := V, \quad \alpha_{p_{u+2}} := m, \quad \text{and as before, } p_{u+1} := \ell.$$

By Lemma 7.4, we see that the cardinality $N_j := |\pi_j^{-1}(U)|$ of the fibre of any $U \in \Lambda_j$ is independent of the choice of U and is given by

$$(28) \quad N_j = \begin{bmatrix} \alpha_{p_{j+1}} - \alpha_{p_j} \\ r_{j+1} - r_j \end{bmatrix}_q q^{p_j(r_{j+1}-r_j)} \quad \text{for } j = 1, \dots, u+1.$$

It follows that

$$(29) \quad |\Lambda_\alpha| = N_{u+1}|\Lambda_{u+1}| = N_{u+1}N_u|\Lambda_u| = \dots = N_{u+1}N_u \dots N_1|\Lambda_1|.$$

Now note that

$$N_{u+1} = \begin{bmatrix} m - \alpha_\ell \\ m - \ell - (\alpha_\ell - \ell) \end{bmatrix}_q q^{\ell(m-\alpha_\ell)} = q^{\ell(m-\alpha_\ell)} \quad \text{and} \quad |\Lambda_1| = |G_{r_1}(A_{p_1})| = \begin{bmatrix} \alpha_{p_1} \\ r_1 \end{bmatrix}_q.$$

Substituting this and (28) in (29) and then comparing with (27), we obtain

$$(30) \quad |\mathfrak{S}_\alpha| = \prod_{j=0}^u \begin{bmatrix} \alpha_{p_{j+1}} - \alpha_{p_j} \\ r_{j+1} - r_j \end{bmatrix}_q q^{p_j(r_{j+1}-r_j)} = \prod_{j=0}^u \begin{bmatrix} \alpha_{p_{j+1}} - \alpha_{p_j} \\ p_{j+1} - p_j \end{bmatrix}_q q^{p_j(r_{j+1}-r_j)},$$

where, as before, we have set $p_0 = 0 = \alpha_{p_0} = r_0$. This implies that $|S_\alpha| = N_\alpha$. The remaining assertions follow from Theorem 5.3 and Corollary 6.2. \square

Remark 7.6. It is clear that if Conjecture 5.6 holds in the affirmative, then N_α given in Theorem 7.5 is precisely the number of minimum weight codewords of $C_\alpha(\ell, m)$. It may be noted that when α is completely consecutive, i.e, when $u = 0$, we have $N_\alpha = (q-1) \begin{bmatrix} \alpha_\ell \\ \ell \end{bmatrix}_q$, which is consistent with the result of Nogin [14] mentioned earlier since in this case $\Omega_\alpha(\ell, m)$ is the Grassmannian $G_\ell(A_\ell)$.

The question as to whether or not the minimum weight codewords of a code generate the code is often of some interest. It is a classical result that this is true in the case of binary Reed-Muller codes (see, e.g., [13, Ch. 13, 6]), whereas for q -ary generalized ReedMuller codes, it is not true, in general (see, e.g., [4, Thm. 1]). For Grassmann codes as well as for related classes of codes such as affine Grassmann codes of an arbitrary level, the minimum weight codewords generate the code (see, e.g., [1, Thm. 18 and Rem. 1]). However, we will show below that Schubert codes are, in general, not generated by their minimum weight codewords.

Theorem 7.7. *Assume that α has more than two consecutive blocks, i.e., $u > 1$. Then the \mathbb{F}_q -linear subspace of $C_\alpha(\ell, m)$ generated by the codewords corresponding to Schubert decomposable elements of $\bigwedge^{m-\ell} V$ is a proper subset of $C_\alpha(\ell, m)$.*

Proof. As in the last proof, let $S_\alpha := \{c_f : f \in \bigwedge^{m-\ell} V \text{ is Schubert decomposable}\}$. The hypothesis on α implies that $p_2 + 1 \leq \ell$, and also that

$$\text{either (i) } \alpha_{p_1} \geq p_1 + 1 \quad \text{or} \quad \text{(ii) } \alpha_{p_1} = p_1 \text{ and } \alpha_{p_2} \geq p_2 + 1.$$

Now fix a basis $\{e_1, \dots, e_m\}$ of V such that $A_i = \langle e_1, \dots, e_{\alpha_i} \rangle$ for $i = 1, \dots, \ell$. Also let $L = \langle e_1, \dots, e_\ell \rangle$ and $g := e_{\ell+1} \wedge \dots \wedge e_m$. Then g is a decomposable

element of $\bigwedge^{m-\ell} V$ such that $c_g(L) \neq 0$. Now suppose $f \in \bigwedge^{m-\ell} V$ is any Schubert decomposable element. Then in case (i) holds, i.e., when $\alpha_{p_1} \geq p_1 + 1$, we find

$$\dim V_f \cap A_{p_1} = \alpha_{p_1} - p_1 \quad \text{and} \quad \dim L \cap A_{p_1} \geq p_1 + 1$$

and consequently, $\dim(V_f \cap L) \geq 1$, which in view of (14) shows that $c_f(L) = 0$. On the other hand, if (ii) holds, then

$$\dim V_f \cap A_{p_2} = \alpha_{p_2} - p_2 \quad \text{and} \quad \dim L \cap A_{p_2} \geq p_2 + 1$$

and consequently, $\dim(V_f \cap L) \geq 1$, which implies once again that $c_f(L) = 0$. It follows that if $c \in C_\alpha(\ell, m)$ is any linear combination of elements of S_α , then $c(L) \neq 0$. Hence $c_g \in C_\alpha(\ell, m)$ is not in the linear span of S_α . \square

Corollary 7.8. *Suppose α is completely non-consecutive and $\ell > 2$. Then $C_\alpha(\ell, m)$ is not generated by its minimum weight codewords.*

Proof. Since α is completely non-consecutive, we have $u = \ell - 1$ and so $u > 1$. Thus the desired result follows from Corollary 6.2 and Theorem 7.7. \square

Remark 7.9. As in Remark 7.6, it is clear that if Conjecture 5.6 holds in the affirmative, then Theorem 7.7 shows that $C_\alpha(\ell, m)$ is not generated by its minimum weight codewords, provided $u > 1$. In fact, our proof of Theorem 7.7 shows that its conclusion as well as the last assertion is also valid when $u = 1$, provided $\alpha_{p_1} > p_1$. On the other hand, when $u = 0$, i.e., when α is consecutive, then $\Omega_\alpha(\ell, m) \simeq G_\ell(A_\ell)$ and $C_\alpha(\ell, m)$ is equivalent to the Grassmann code $C(\ell, \alpha_\ell)$. So we know from the work of Nogin [14] that $C_\alpha(\ell, m)$ is generated by its minimum weight codewords. Moreover, when $u = 1$ and $\alpha_{p_1} = p_1$, then any $W \in \Omega_\alpha(\ell, m)$ satisfies $W \cap A_{p_1} = A_{p_1}$, i.e., $A_{p_1} \subseteq W$, and hence $W \mapsto W/A_{p_1}$ sets up a natural isomorphism between $\Omega_\alpha(\ell, m)$ and $G_{\ell-p_1}(A_\ell/A_{p_1})$. Consequently, $C_\alpha(\ell, m)$ is equivalent to the Grassmann code $C(\ell - p_1, \alpha_\ell - \alpha_{p_1})$. So once again, Nogin's result implies that $C_\alpha(\ell, m)$ is generated by its minimum weight codewords in this case.

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DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY BOMBAY,
POWAI, MUMBAI 400076, INDIA.
E-mail address: srg@math.iitb.ac.in

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY BOMBAY,
POWAI, MUMBAI 400076, INDIA.
E-mail address: psinghprasant@gmail.com